Exercise. Let $V = \mathbb{R}^2$. Show that the forms $x_1 x_2$ and $2x_1^2 - 2x_2^2$ on V are equivalent.

Solution.

Let $\qquad \psi(x_1, x_2) = 2x_1^2 - 2x_2^2 \qquad$ and $\qquad \phi(x_1, x_2) = x_1 x_2$

ψ and ϕ are equivalent $\iff \exists M \in GL(2, \mathbb{R})$ s.t. $\psi(x_1, x_2) = \phi\left(M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$

$$\begin{aligned}
\psi(x_1, x_2) &= 2x_1^2 - 2x_2^2 \\
&= 2(x_1 - x_2)(x_1 + x_2) \\
&= \phi(2x_1 - 2x_2, x_1 + x_2) \\
&= \phi\left(\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)
\end{aligned}$$

Thus, $x_1 x_2$ and $2x_1^2 - 2x_2^2$ are equivalent