

**Rutgers University: Algebra Written Qualifying Exam**  
**August 2014: Problem 4 Solution**

**Exercise.** Let  $V = \mathbb{R}^2$ . Show that the forms  $x_1x_2$  and  $2x_1^2 - 2x_2^2$  on  $V$  are equivalent.

Solution.

Let  $\psi(x_1, x_2) = 2x_1^2 - 2x_2^2$  and  $\phi(x_1, x_2) = x_1x_2$

$\psi$  and  $\phi$  are equivalent  $\iff \exists M \in GL(2, \mathbb{R})$  s.t.  $\psi(x_1, x_2) = \phi\left(M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$

$$\begin{aligned}\psi(x_1, x_2) &= 2x_1^2 - 2x_2^2 \\ &= 2(x_1 - x_2)(x_1 + x_2) \\ &= \phi(2x_1 - 2x_2, x_1 + x_2) \\ &= \phi\left(\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)\end{aligned}$$

Thus,  $x_1x_2$  and  $2x_1^2 - 2x_2^2$  are equivalent